

# Synchronization of a Nonlinear Dynamical System with Fractional Orders

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**Abstract:** In this paper, the dynamics of a new fractional-order system is investigated. Firstly, the numerical solutions of the system by the improved predictor-corrector algorithm are obtained. Based on this, the dynamics of the system is analyzed by numerical simulations. A chaotic attractor and periodic orbits with different values of the derivative order or the system parameter are demonstrated.

## 1. Introduction

In 1990, chaos synchronization was presented by Pecora and Carroll [1] firstly. Nowadays, there are several different types of synchronization, such as lag synchronization [2], phase synchronization [3], mixed synchronization [4], projective synchronization (PS) [5] and function projective synchronization (FPS) [6-7], etc. Meanwhile, it is well known that FPS is characterized that the drive and response systems could be synchronized up to a scaling function, but not a constant. It is obvious that the unpredictability of the scaling function in FPS can additionally enhance the security of communication.

## 2. The scheme of FPS

Firstly, the following fractional-order system is taken as the drive system

$$\frac{d^{q_d} \mathbf{x}}{dt^{q_d}} = \mathbf{f}(\mathbf{x}), \quad (1)$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in R^n$  is the state vector of system (1),  $\mathbf{f}(\mathbf{x}) = (f_1(x), f_2(x), \dots, f_n(x))^T$  is a continuous nonlinear vector function,  $\mathbf{q}_d = (q_{d1}, q_{d1}, \dots, q_{dn})^T$  is the vector of derivative orders with the  $0 < q_i < 1, (1 \leq i \leq n)$ .

The corresponding response system is defined as

$$\frac{d^{q_r} \mathbf{y}}{dt^{q_r}} = \mathbf{g}(\mathbf{y}) + \mathbf{u}(\mathbf{x}, \mathbf{y}), \quad (2)$$

where  $\mathbf{y} = (y_1, y_2, \dots, y_m)^T \in R^m$  is the state vector of response system (2). Meanwhile,  $\mathbf{u}(\mathbf{x}, \mathbf{y})$  is the controller for the function projective synchronization of the drive system (1) and the response system (2). For simplicity, we assume that  $m \leq n$ .

The synchronization error vector is defined as  $\mathbf{e} = \mathbf{y} - \mathbf{v}(\mathbf{x})\mathbf{x}$ , the scaling function  $v_i(x) (i = 1, 2, \dots, m)$ , where  $\mathbf{v}(\mathbf{x}) = \text{diag}(v_1(x), v_2(x), \dots, v_m(x))^T$  and is the continuously differentiable and limited boundary. The function projective synchronization is realized if there exist a vector function  $\mathbf{v}(\mathbf{x})$  such that

$$\lim_{t \rightarrow \infty} \|\mathbf{e}\| = \lim_{t \rightarrow \infty} \|\mathbf{y} - \mathbf{v}(\mathbf{x})\mathbf{x}\| = 0. \quad (3)$$

If  $\mathbf{v}(\mathbf{x}) = \mathbf{I}$ , then the synchronization scheme becomes the complete synchronization, and the anti-synchronization for  $\mathbf{v}(\mathbf{x}) = -\mathbf{I}$ , where  $\mathbf{I}$  is a  $m \times m$  identify matrix. If  $\mathbf{v}(\mathbf{x}) = c\mathbf{I}$ ,  $c$  is an arbitrary real constant, then the projective synchronization will be realized between the drive and the response systems.

In order to realize the function projective synchronization of drive and response systems, a compensation controller is defined as

$$\boldsymbol{\theta}(\mathbf{x}) = \frac{d^{q_r}(\mathbf{v}(\mathbf{x})\mathbf{x})}{dt^{q_r}} - \mathbf{g}(\mathbf{v}(\mathbf{x})\mathbf{x}), \quad (4)$$

then the synchronization controller is designed as

$$\mathbf{u}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\theta}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}, \mathbf{y}), \quad (5)$$

where  $\boldsymbol{\tau}(\mathbf{x}, \mathbf{y})$  is a vector function.

When the compensation controller (4) and synchronization controller (5) are substituted into the system (2), the response system (1) can be rewritten as in the following form

$$\frac{d^{q_r} \mathbf{e}}{dt^{q_r}} = \mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{v}(\mathbf{x})\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) = \mathbf{A}(\mathbf{x}, \mathbf{y})\mathbf{e} + \boldsymbol{\tau}(\mathbf{x}, \mathbf{y}), \quad (6)$$

where  $\mathbf{A}(\mathbf{x}, \mathbf{y})\mathbf{e} = \mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{v}(\mathbf{x})\mathbf{x})$ , and  $\boldsymbol{\tau}(\mathbf{x}, \mathbf{y})$  is a  $m \times m$  matrix.

**Theorem 1.** Given the fractional-order system (1), there exists a control vector  $\boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) = \mathbf{B}(\mathbf{x}, \mathbf{y})\mathbf{e}$  such that the function projective between (1) and (2) can be achieved if

$$\mathbf{P}[\mathbf{A}(\mathbf{x}, \mathbf{y}) + \mathbf{B}(\mathbf{x}, \mathbf{y})] + [\mathbf{A}(\mathbf{x}, \mathbf{y}) + \mathbf{B}(\mathbf{x}, \mathbf{y})]^H \mathbf{P} = -\mathbf{Q}, \quad (7)$$

where  $\mathbf{P}$  and  $\mathbf{Q}$  are real symmetric positive define matrix, and  $\mathbf{B}(\mathbf{x}, \mathbf{y})$ , and  $H$  stands for conjugate transpose of a matrix.

### 3. Numerical simulations

In this section, we will take the following fractional-order system as the drive system, and the FPS for the system is investigated. Meanwhil, we will consider the drive system (8) in the commensurate case, namely, all the derivative orders are taken as  $q_d = q_1 = q_2 = q_3 = 0.99$ . The derive system is written as the following form:

$$\begin{cases} D^{q_d} x_1 = ax_1 - x_2 \\ D^{q_d} x_2 = x_1 - x_3 - bx_2 \\ D^{q_d} x_3 = ax_1 + 4x_3(x_2 - 2) \end{cases}, \quad (8)$$

where  $\mathbf{x} = (x_1, x_2, x_3)^T$  is the vector of the variables of the above system, and  $q_d$  is the derivative order.

The response system can be described by the following fractional differential equations

$$\begin{cases} D^{q_r} y_1 = ay_1 - y_2 \\ D^{q_r} y_2 = y_1 - y_3 - by_2 \\ D^{q_r} y_3 = ay_1 + 4y_3(y_2 - 2) \end{cases}. \quad (9)$$

where  $\mathbf{y} = (y_1, y_2, y_3)^T$  is the vector of the variables of the response system, and  $q_r$  is the derivative order. The corresponding response system with controller is

$$\begin{bmatrix} D^\beta y_1 \\ D^\beta y_2 \\ D^\beta y_3 \end{bmatrix} = \begin{bmatrix} a(y_1 - y_2) \\ y_1 - y_3 - by_1 \\ ay_1 + 4y_3(y_1 - 2) \end{bmatrix} + u(x, y). \quad (10)$$

Defined the synchronization error vector as

$$\mathbf{e} = (e_1, e_2, e_3) = (y_1 - \kappa_1(x)x_1, y_2 - \kappa_2(x)x_2, y_3 - \kappa_3(x)x_3). \quad (11)$$

where  $\kappa(x) = \text{diag}(\kappa_1(x), \kappa_2(x), \kappa_3(x))$ .

By computation, we can get  $\mathbf{A}(\mathbf{x}, \mathbf{y})$  is following

$$\mathbf{A}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} a & -1 & 0 \\ 1 & -b & -1 \\ a & 4\kappa_3(\mathbf{x})x_3 & -8 + 4y_1 \end{bmatrix}. \quad (12)$$

Using the theorem, the matrix  $\mathbf{B}(\mathbf{x}, \mathbf{y})$  is chosen as

$$\mathbf{B}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} -2a & 0 & -a \\ 0 & 0 & 1 - 4\kappa_3(\mathbf{x})x_3 \\ 0 & 0 & -4y_1 \end{bmatrix}. \quad (13)$$

Hence, the error dynamical system is

$$\begin{bmatrix} D^{q_r} e_1 \\ D^{q_r} e_2 \\ D^{q_r} e_3 \end{bmatrix} = \begin{bmatrix} -a & -1 & -a \\ 1 & -b & -4\kappa_3(\mathbf{x})x_3 \\ a & 4\kappa_3(\mathbf{x})x_3 & -8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \quad (14)$$

Choose a real symmetric positive definite matrix  $\mathbf{P} = \text{diag}(1, 1, 1)$ , then we can obtain

$$\mathbf{P}[\mathbf{A}(\mathbf{x}, \mathbf{y}) + \mathbf{B}(\mathbf{x}, \mathbf{y})] + [\mathbf{A}(\mathbf{x}, \mathbf{y}) + \mathbf{B}(\mathbf{x}, \mathbf{y})]^H \mathbf{P} = \text{diag}(-2a, -2b, -16). \quad (15)$$

Therefore, another real symmetric positive definite matrix is chosen as  $\mathbf{Q} = \text{diag}(2a, -4, -16)$ , the expression (34) can be written as

$$\mathbf{P}[\mathbf{A}(\mathbf{x}, \mathbf{y}) + \mathbf{B}(\mathbf{x}, \mathbf{y})] + [\mathbf{A}(\mathbf{x}, \mathbf{y}) + \mathbf{B}(\mathbf{x}, \mathbf{y})]^H \mathbf{P} = -\mathbf{Q}. \quad (16)$$

According to the theorem, the FPS of drive system (8) and response system (9) is achieved.

In numerical simulations, we choose  $\kappa(x) = \text{diag}(1 + 0.1x_1x_3, x_2 + x_1, 1 + x_3)$ . The simulation results are shown in the Figs.5, from which it can be seen that the error variables  $e_1, e_2$ , and  $e_3$  converge to zero as  $t \rightarrow 20s$ , which implies the FPS between the drive and response systems is achieved under the controller.

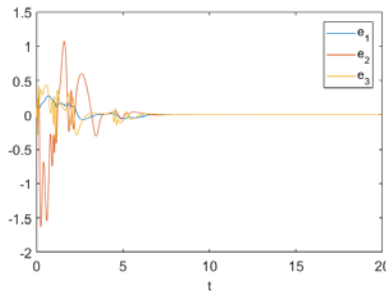


Fig.1. The FPS results between the drive and the response systems

#### **4. Summary**

In this paper, a new system with fractional order with a one-scroll chaotic attractor is proposed. Firstly, the numerical solutions of the system by the improved predictor-corrector algorithm are obtained. The dynamics of the system, including chaotic attractor and periodic orbits, is analyzed by numerical simulations.

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